Accepted Manuscript

A compact difference scheme for numerical solutions of second order dual-phase-lagging models of microscale heat transfer

M.A. Castro, F. Rodríguez, J. Cabrera, J.A. Martín

PII: S0377-0427(14)00477-4
DOI: http://dx.doi.org/10.1016/j.cam.2014.11.006
Reference: CAM 9848

To appear in: *Journal of Computational and Applied Mathematics*

Received date: 14 October 2014
Revised date: 26 October 2014

Please cite this article as: M.A. Castro, F. Rodríguez, J. Cabrera, J.A. Martín, A compact difference scheme for numerical solutions of second order dual-phase-lagging models of microscale heat transfer, *Journal of Computational and Applied Mathematics* (2014), http://dx.doi.org/10.1016/j.cam.2014.11.006

A compact difference scheme for numerical solutions of second order dual-phase-lagging models of microscale heat transfer

M.A. Castro, F. Rodríguez, J. Cabrera, J.A.Martín*

Dep. Matemática Aplicada, Universidad de Alicante, Apdo. 99, 03080 Alicante, Spain.

Abstract

Dual-phase-lagging (DPL) models constitute a family of non-Fourier models of heat conduction that allow for the presence of time lags in the heat flux and the temperature gradient. These lags may need to be considered when modeling microscale heat transfer, and thus DPL models have found application in the last years in a wide range of theoretical and technical heat transfer problems. Consequently, analytical solutions and methods for computing numerical approximations have been proposed for particular DPL models in different settings.

In this work, a compact difference scheme for second order DPL models is developed, providing higher order precision than a previously proposed method. The scheme is shown to be unconditionally stable and convergent, and its accuracy is illustrated with numerical examples.

Keywords: Non-Fourier heat conduction, DPL models, Finite differences, Convergence and stability.

2010 MSC: 35R10, 65M06, 65M12

*Corresponding author.

Email addresses: ma.castro@ua.es (M.A. Castro), f.rodriguez@ua.es (F. Rodríguez), jesus.cabrera@ua.es (J. Cabrera), jose.martin@ua.es (J.A.Martín)

1. Introduction

Technical advances in nanomaterials and in the applications of ultrafast lasers have lead in last two decades to an increasing interest in non-Fourier models of heat conduction [1, 2, 3, 4, 5]. These models try to account for phenomena, such as finite speeds of propagation and wave behaviors, that appear when studying heat transfer at the microscale level, i.e., in very short time intervals or at very small space dimensions [6, 7].

The basis for the dual-phase-lagging (DPL) family of models is the introduction of two time lags into the Fourier law [7, 8, 9],

$$\mathbf{q}(\mathbf{r}, t + \tau_q) = -k\nabla T(\mathbf{r}, t + \tau_T), \tag{1}$$

where τ_q and τ_T are, respectively, the phase lags in the heat flux vector, $\mathbf{q}$, and the temperature gradient, ∇T, t and $\mathbf{r}$ are the time and spatial coordinates, and k is the conductivity.

When both lags are zero, so that the classical Fourier law is recovered, the combination of (1) with the conservation of energy principle leads to the diffusion or classical heat conduction equation. Otherwise, a partial differential equation with delay is obtained [10, 11].

Most commonly, though, first or higher order approximations in the time lags in (1) are used [8, 12, 13]. In this work, the heat equation resulting from first order approximations,

$$\frac{\partial}{\partial t}T(\mathbf{r}, t) + \tau_q \frac{\partial^2}{\partial t^2}T(\mathbf{r}, t) = \alpha \left(\Delta T(\mathbf{r}, t) + \tau_T \Delta \frac{\partial}{\partial t} T(\mathbf{r}, t) \right), \tag{2}$$

usually referred to as the DPL model [8], will be denoted DPL(1,1), and the equations derived from second order approximation in τ_q and up to second order approximation in τ_T will be denoted DPL(2,1),

$$\frac{\partial}{\partial t}T(\mathbf{r},t)+\tau_q\frac{\partial^2}{\partial t^2}T(\mathbf{r},t)+\frac{\tau_q^2}{2}\frac{\partial^3}{\partial t^3}T(\mathbf{r},t)=\alpha\left(\Delta T(\mathbf{r},t)+\tau_T\Delta\frac{\partial}{\partial t}T(\mathbf{r},t)\right), \tag{3}$$

and DPL(2,2),

$$\frac{\partial}{\partial t}T(\mathbf{r},t)+\tau_q\frac{\partial^2}{\partial t^2}T(\mathbf{r},t)+\frac{\tau_q^2}{2}\frac{\partial^3}{\partial t^3}T(\mathbf{r},t)=\alpha\left(\Delta T(\mathbf{r},t)+\tau_T\Delta\frac{\partial}{\partial t}T(\mathbf{r},t)+\frac{\tau_T^2}{2}\Delta\frac{\partial^2}{\partial t^2}T(\mathbf{r},t)\right). \tag{4}$$

The construction of numerical solutions for DPL(1,1) model and variations in different settings has been addressed in previous works (e.g., [14, 15, 16, 17, 18, 19]). For DPL(2,2) models, a Crank-Nicholson type difference scheme was presented in [20]. The objective of this work is to develop a higher order, compact difference scheme for the same problem considered in [20], in an similar way as was done in [17] for DPL(1,1) models but employing a more direct approach to construct the scheme and prove its stability and convergence.

As in [20], a general equation for heat conduction in one dimension will be considered,

$$\frac{\partial}{\partial t}\left(AT\left(x,t\right)+B\frac{\partial}{\partial t}T\left(x,t\right)+C\frac{\partial^2}{\partial t^2}T\left(x,t\right)\right)=\frac{\partial^2}{\partial x^2}\left(T\left(x,t\right)+D\frac{\partial}{\partial t}T\left(x,t\right)+E\frac{\partial^2}{\partial t^2}T\left(x,t\right)\right), \tag{5}$$

which includes DPL(2,2), as given by (4), by taking

$$A=\frac{1}{\alpha},\quad B=\frac{\tau_q}{\alpha},\quad C=\frac{\tau_q^2}{2\alpha},\quad D=\tau_T,\quad E=\frac{\tau_T^2}{2}, \tag{6}$$

and reduces to DPL(2,1) when $E=0$. The problem is stated for a finite domain $x\in[0,l]$, with Dirichlet boundary conditions,

$$T(0,t)=T(l,t)=0,\ t\geq 0, \tag{7}$$

and initial conditions

$$T(x,0)=\phi(x),\ \frac{\partial}{\partial t}T(x,0)=\varphi(x),\ \frac{\partial^2}{\partial t^2}T(x,0)=\psi(x),\ x\in[0,l]\,. \tag{8}$$

The rest of the paper is organized as follows. In the next section, the new compact difference scheme for DPL(2,2) model is constructed. In Section 3, after expressing the method as a two-level scheme, the unconditional stability of the method is proved. Next, in Section 4, assuming sufficient regularity of the solution, the consistency of the method is shown, and bounds on the truncation errors are obtained. In the last section, numerical examples are presented, illustrating the higher accuracy of the new method in comparison with the scheme previously proposed in [20].

2. Construction of the compact finite difference scheme

First, two new variables, $v(x,t)$ and $u(x,t)$, are introduced in order to express (5) as a first order system in t,

$$v\left(x,t\right)=BT\left(x,t\right)+C\frac{\partial}{\partial t}T\left(x,t\right), \tag{9}$$

and

$$u\left(x,t\right)=AT\left(x,t\right)+\frac{\partial}{\partial t}v\left(x,t\right). \tag{10}$$

Thus, using (9) and (10), it can be shown that Eq. 5 can be written as

$$\frac{\partial}{\partial t}u\left(x,t\right)=a\frac{\partial^2}{\partial x^2}T\left(x,t\right)+b\frac{\partial^2}{\partial x^2}v\left(x,t\right)+c\frac{\partial^2}{\partial x^2}u\left(x,t\right), \tag{11}$$

where

$$a=(C^2+EB^2-BDC-ACE)/C^2,\ b=(DC-BE)/C^2,\ c=E/C. \tag{12}$$

Consequently, writing (9) and (10) in the form

$$\frac{\partial}{\partial t}T\left(x,t\right)=\frac{1}{C}\left(v\left(x,t\right)-BT\left(x,t\right)\right) \tag{13}$$

and

$$\frac{\partial}{\partial t}v\left(x,t\right)=u\left(x,t\right)-AT\left(x,t\right), \tag{14}$$

an equivalent problem to that consisting of equation (5), boundary conditions (7), and initial conditions (8) is given by the system of equations (11)-(14), with boundary conditions

$$u(0,t)=u(l,t)=0,\ T(0,t)=T(l,t)=0,\ v(0,t)=v(l,t)=0,\ t\geq 0, \tag{15}$$

and initial conditions

$$u\left(x,0\right)=A\phi\left(x\right)+B\varphi\left(x\right)+C\psi\left(x\right),\ T\left(x,0\right)=\phi\left(x\right),\ v\left(x,0\right)=B\phi\left(x\right)+C\varphi\left(x\right),\ x\in\left[0,l\right]. \tag{16}$$

Numerical solutions will be computed inside a bounded domain $[0,l]\times[0,T_M]$, for some fixed $T_M>0$, at the points of the mesh $\{(x_j,t_n),j=0\dots P,n=0\dots N\}$, where $l=Ph$ and $T_M=Nk$ for spatial increment $h=\Delta x$ and temporal increment $k=\Delta t$. In what follows, for any function w, w_j^n will denote its value at the point (x_j,t_n).

To derive the new numerical scheme for DPL(2,2) model, the following finite difference approximations will be used in equations (13) and (14),

$$\begin{aligned}\frac{1}{k}\left(T_j^{n+1}-T_j^n\right) &= -\frac{B}{2C}\left(T_j^{n+1}+T_j^n\right)+\frac{1}{2C}\left(v_j^{n+1}+v_j^n\right), &(17)\\ \frac{1}{k}\left(v_j^{n+1}-v_j^n\right) &= \frac{1}{2}\left(u_j^{n+1}+u_j^n\right)-\frac{A}{2}\left(T_j^{n+1}+T_j^n\right). &(18)\end{aligned}$$

For equation (13), two different approaches may be followed. Writing

$$f\left(x,t\right)=\frac{\partial^2}{\partial x^2}T\left(x,t\right),\quad g\left(x,t\right)=\frac{\partial^2}{\partial x^2}v\left(x,t\right),\quad s\left(x,t\right)=\frac{\partial^2}{\partial x^2}u\left(x,t\right),$$

using in (13) the finite approximations

$$\frac{1}{k}\left(u_j^{n+1}-u_j^n\right)=\frac{a}{2}\left(f_j^{n+1}+f_j^n\right)+\frac{b}{2}\left(g_j^{n+1}+g_j^n\right)+\frac{c}{2}\left(s_j^{n+1}+s_j^n\right),$$

and considering for the functions f, g, and s the compact fourth order finite difference approximation [21],

$$\frac{1}{10} f_{j-1}^n + f_j^n + \frac{1}{10} f_{j+1}^n = \frac{6}{5} \delta_x^2 T_j^n,$$

where

$$\delta_x^2 w_j^n = \frac{1}{h^2} \left(w_{j-1}^n - 2w_j^n + w_{j+1}^n \right),$$

with analogous approximations for g and s, after some algebraic manipulations one can get to the expression

$$\begin{aligned} & u_{j-1}^{n+1} + 10u_j^{n+1} + u_{j+1}^{n+1} - \left(u_{j-1}^n + 10u_j^n + u_{j+1}^n \right) = \\ & 6ak\delta_x^2 \left(T_j^{n+1} + T_j^n \right) + 6bk\delta_x^2 \left(v_j^{n+1} + v_j^n \right) + 6ck\delta_x^2 \left(u_j^{n+1} + u_j^n \right). \end{aligned} \tag{19}$$

This approach, analogous to that used in [17] for the corresponding DPL(1,1) model, emphasizes the obtention of the scheme from the class of compact finite approximations presented in [21], suggesting the way for generalizations by using other higher-order compact finite difference from [21].

However, a much more direct derivation of (19) can be obtained by using for equation (13) the following finite difference approximations,

$$\begin{aligned} & \frac{1}{12k} \left(u_{j+1}^{n+1} - u_{j+1}^n \right) + \frac{5}{6k} \left(u_j^{n+1} - u_j^n \right) + \frac{1}{12k} \left(u_{j-1}^{n+1} - u_{j-1}^n \right) = \\ & \frac{a}{2} \delta_x^2 \left(T_j^{n+1} + T_j^n \right) + \frac{b}{2} \delta_x^2 \left(v_j^{n+1} + v_j^n \right) + \frac{c}{2} \delta_x^2 \left(u_j^{n+1} + u_j^n \right), \end{aligned}$$

which are analogous to those leading to a classical method for the diffusion equation [22, p. 191], and let immediately to equation (19). Besides its simpler derivation, the advantage of this approach is that it relates the new scheme to a difference method for classical heat conduction that is known to be unconditionally stable, providing the basis for a direct analysis of the stability of the new method in the next section.

Thus, the new scheme is defined by equations (17), (18), and (19), together with the discretized boundary and initial conditions, corresponding to (15) and (16),

$$u_0^n = u_P^n,\ T_0^n = T_P^n = 0,\ v_0^n = v_P^n = 0,\ n = 0 \dots N, \tag{20}$$

and

$$u_j^0 = A\phi(jh) + B\varphi(jh) + C\psi(jh),\ T_j^0 = \phi(jh),\ v_j^0 = B\phi(jh) + C\varphi(jh),\ j = 0, \dots, P. \tag{21}$$

Next, the scheme will be expressed in matrix form. Consider, for $n = 0 \dots N$, the vectors U^n, T^n, and V^n, where U^n, respectively T^n and V^n, stacks up the $P-1$ values u_j^n, respectively T_j^n and v_j^n, for $j = 1, \dots, P-1$. Writing $r = h/k^2$, and introducing the two tridiagonal $(P-1) \times (P-1)$ matrices $M = tridiag\,(1, -2, 1)$ and $S = tridiag\,(1, 10, 1)$, i.e., tridiagonal matrices with ones in the upper and lower diagonals and with -2, for M, or with 10, for S, in the main diagonal, the expression of the scheme as a system of matrix equations is given by

$$S\left(U^{n+1} - U^n\right) = 6r\left(aM\left(T^{n+1} + T^n\right) + bM\left(V^{n+1} + V^n\right) + cM\left(U^{n+1} + U^n\right)\right), \tag{22}$$

$$\frac{1}{k}\left(T^{n+1} - T^n\right) = -\frac{B}{2C}\left(T^{n+1} + T^n\right) + \frac{1}{2C}\left(V^{n+1} + V^n\right), \tag{23}$$

and

$$\frac{1}{k}\left(V^{n+1} - V^n\right) = \frac{1}{2}\left(U^{n+1} + U^n\right) - \frac{A}{2}\left(T^{n+1} + T^n\right). \tag{24}$$

More practical expressions can be obtained by solving for T^{n+1} and V^{n+1} in (23) and (24), and substituting into (22). After some rearrangements, and writing

$$x_k = 4C + 2Bk + Ak^2,\ y_k = 4C - 2Bk - Ak^2,\ z_k = 4C + 2Bk - Ak^2, \tag{25}$$

and

$$\alpha_k = r\left(c + \frac{x_k + z_k}{x_k}\frac{bk}{4} + \frac{ak^2}{x_k}\right),\ \beta_k = r\left(a + \frac{ay_k}{x_k} - \frac{x_k + y_k}{x_k}\frac{bAk}{2}\right),\ \gamma_k = r\left(b + \frac{4ak}{x_k} + \frac{bz_k}{x_k}\right), \tag{26}$$

the following expressions can be obtained,

$$(S - 6\alpha_k M)\,U^{n+1} = (S + 6\alpha_k M)\,U^n + 6\beta_k M T^n + 6\gamma_k M V^n, \tag{27}$$

$$T^{n+1} = \frac{y_k}{x_k}T^n + \frac{4k}{x_k}V^n + \frac{k^2}{x_k}U^n + \frac{k^2}{x_k}U^{n+1}, \tag{28}$$

$$V^{n+1} = -\frac{4kAC}{x_k}T^n + \frac{z_k}{x_k}V^n + \frac{2C + Bk}{x_k}kU^n + \frac{2C + Bk}{x_k}kU^{n+1}, \tag{29}$$

which show that, using the initial values given by (21) and the boundary conditions (20), the application of the method to obtain the numerical approximations corresponding to the next step time $n + 1$ consists of solving for U^{n+1} in (27) and substituting into equations (28) and (29) to compute T^{n+1} and V^{n+1}. Note that the coefficient of U^{n+1} in (27) is a tridiagonal matrix, which allows for a very efficient computation of U^{n+1}.

Pseudocode of the algorithm used to implement the new scheme is presented in Table 1.

3. Unconditional stability of the method

To analyze the stability of the method, first it will be written as a two-level scheme. Then, using a limit argument and a form of Gronwall lemma, it will be shown that the powers of the matrix of the scheme are uniformly bounded, and hence the unconditional stability of the method will follow.

Writing

$$X_k = I - 6\alpha_k S^{-1}M, \quad Y_k = I + 6\alpha_k S^{-1}M,$$

Input data

input DPL problem parameters, α, τ_T, τ_q, l, and functions giving the initial conditions, ϕ, φ, and ψ

set $A = 1/\alpha$, $B = \tau_q/\alpha$, $C = \tau_q^2/2\alpha$, $D = \tau_T$, $E = \tau_T^2/2$

input domain and mesh parameters, T_M, h, and k

set $P = l/h$, $N = T_M/k$, $r = k/h^2$

Compute auxiliary coefficients and matrices

compute a, b, and c (Eq. 12), and x_k, y_k, and z_k (Eq. 25)

compute α_k, β_k, and γ_k (Eq. 26)

compute tridiagonal matrices $M = tridiag\,(1, -2, 1)$ and $S = tridiag\,(1, 10, 1)$

Initial values (Eq. 21)

for $j = 1$ to $P - 1$ do $T_j^0 = \phi(jh)$; $V_j^0 = BT_j^0 + C\varphi(jh)$; $U_j^0 = AT_j^0 + B\varphi(jh) + C\psi(jh)$; end for

Advance computation to next time step until done (saving partial computations and reusing memory)

for $n = 1$ to N do

Compute vector U^1 by solving the tridiagonal system given by Eq. 27

$(S - 6\alpha_k M)\,U^1 = (S + 6\alpha_k M)\,U^0 + 6\beta_k M T^0 + 6\gamma_k M V^0$

Compute vectors V^1 and T^1 (Eqs. 28 and 29)

$T^1 = \frac{y_k}{x_k}T^0 + \frac{4k}{x_k}V^0 + \frac{k^2}{x_k}U^0 + \frac{k^2}{x_k}U^1$

$V^1 = -\frac{4kAC}{x_k}T^0 + \frac{z_k}{x_k}V^0 + \frac{2C+Bk}{x_k}kU^0 + \frac{2C+Bk}{x_k}kU^1$

Save n-step numerical approximation, T^1

Reuse memory, $T^0 = T^1$, $V^0 = V^1$, $U^0 = U^1$

end for

Table 1: Pseudocode of the algorithm for the new difference scheme

where I denotes the $(P-1)\times(P-1)$ identity matrix, from (27) one gets,

$$U^{n+1} = X_k^{-1}Y_kU^n + 6\beta_kX_k^{-1}S^{-1}MT^n + 6\gamma_kX_k^{-1}S^{-1}MV^n,$$

and, from (28) and (29),

$$\begin{aligned} T^{n+1} &= \frac{2k^2}{x_k}X_k^{-1}U^n + \frac{y_k}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k - \beta_k\frac{k^2}{y_k}\right)S^{-1}M\right)T^n \\ &\quad + \frac{4k}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k - \frac{1}{4}\gamma_k k\right)S^{-1}M\right)V^n, \end{aligned}$$

and

$$\begin{aligned} V^{n+1} &= \frac{2\left(2C+Bk\right)k}{x_k}X_k^{-1}U^n - \frac{4kAC}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k + \beta_k\frac{(2C+Bk)}{4AC}\right)S^{-1}M\right)T^n \\ &\quad + \frac{z_k}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k - \gamma_k\frac{(2C+Bk)\,k}{z_k}\right)S^{-1}M\right)V^n. \end{aligned}$$

Thus, introducing the stack vector $[U^n, T^n, V^n]^T$, with superindex T denoting the transpose, the method can be written as the two-level scheme

$$\left[U^{n+1}, T^{n+1}, V^{n+1}\right]^T = Q_k\left[U^n, T^n, V^n\right]^T, \tag{30}$$

where the matrix of the scheme, Q_k, is given by

$$Q_k = \begin{bmatrix} X_k^{-1}Y_k & 6\beta_kX_k^{-1}S^{-1}M & 6\gamma_kX_k^{-1}S^{-1}M \\ \frac{2k^2}{x_k}X_k^{-1} & \frac{y_k}{x_k}X_k^{-1}\left(I-6\left(\alpha_k-\beta_k\frac{k^2}{y_k}\right)S^{-1}M\right) & Q_k[2,3] \\ \frac{2(2C+Bk)k}{x_k}X_k^{-1} & -\frac{4kAC}{x_k}X_k^{-1}\left(I-6\left(\alpha_k+\beta_k\frac{(2C+Bk)}{4AC}\right)S^{-1}M\right) & Q_k[3,3] \end{bmatrix}, \tag{31}$$

with

$$Q_k[2,3] = \frac{4k}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k - \frac{1}{4}\gamma_k k\right)S^{-1}M\right),$$

and

$$Q_k[3,3] = \frac{z_k}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k - \gamma_k\frac{(2C+Bk)\,k}{z_k}\right)S^{-1}M\right).$$

Writing

$$X^{-1} = \left(I - 6rcS^{-1}M\right)^{-1}, \quad Y = I + 6rcS^{-1}M,$$

it can be shown that the matrix $Q = \lim_{k\to 0} Q_k$ is given by

$$Q = \begin{bmatrix} X^{-1}Y & 12raX^{-1}S^{-1}M & 12rbX^{-1}S^{-1}M \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then, since the matrices M, S, X, and Y commute, one gets the following expression for the powers of the limit matrix Q,

$$Q^n = \begin{bmatrix} \left(X^{-1}Y\right)^n & -\frac{a}{c}\left(I - \left(X^{-1}Y\right)^n\right) & -\frac{b}{c}\left(I - \left(X^{-1}Y\right)^n\right) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Here, the matrix $X^{-1}Y$ is similar to the matrix of an unconditionally stable and convergent scheme for the classical diffusion equation [22, p. 191, scheme 12], and so its powers are uniformly bounded, i.e., there are constants $k_0 > 0$ and $C > 0$ such that, for $0 < k < k_0$ and $0 \leq nk \leq T_M$, it holds that

$$\|Q^n\| \leq C.$$

It can be checked, by direct computation, that $\|Q_k - Q\| = O(k)$, so that there is a constant $\alpha > 0$ such that, for $0 < k < k_0$,

$$\|Q_k - Q\| \leq \alpha k.$$

Then, writing

$$\begin{aligned} Q_k^n - Q^n &= \sum_{j=0}^{n-1} Q_k^{n-1-j} \left(Q_k - Q\right) Q^j \\ &= \sum_{j=0}^{n-1} \left(Q_k^{n-1-j} - Q^{n-1-j} + Q^{n-1-j}\right) \left(Q_k - Q\right) Q^j \\ &= \sum_{j=0}^{n-1} \left(Q_k^{n-1-j} - Q^{n-1-j}\right) \left(Q_k - Q\right) Q^j + \sum_{j=0}^{n-1} Q^{n-1-j} \left(Q_k - Q\right) Q^j, \end{aligned}$$

it follows that

$$\begin{aligned} \|Q_k^n - Q^n\| &\leq \sum_{j=0}^{n-1} \left\|Q_k^{n-1-j} - Q^{n-1-j}\right\| \|Q_k - Q\| \left\|Q^j\right\| + \sum_{j=0}^{n-1} \left\|Q^{n-1-j}\right\| \|Q_k - Q\| \left\|Q^j\right\| \\ &\leq C\alpha k \sum_{j=0}^{n-1} \left\|Q_k^{n-1-j} - Q^{n-1-j}\right\| + nC^2\alpha k \\ &\leq C\alpha k \sum_{j=0}^{n-1} \left\|Q_k^j - Q^j\right\| + C^2\alpha T_M. \end{aligned}$$

From the last inequality, a uniform bound for $\|Q_k^n - Q^n\|$ can be obtained, by using the form of Gronwall lemma stated next [23, pp. 186].

Lemma 1 (Gronwall inequality)**.** *Let $u(n)$, $f(n)$, p, and q be nonnegative such that*

$$u\left(n\right) \leq p + q \sum_{j=a}^{n-1} f\left(j\right) u\left(j\right), \quad n \geq a,$$

then

$$u\left(n\right) \leq p \prod_{j=a}^{n-1} \left(1 + qf\left(j\right)\right), \quad n \geq a.$$

Thus, taking

$$u\left(n\right) = \|Q_k^n - Q^n\|,\ p = C^2\alpha T_M,\ q = C\alpha k,\ a = 0,\ f\left(j\right) = 1,$$

one gets, for all $n \geq 0$,

$$\|Q_k^n - Q^n\| \leq C^2\alpha T_M \left(1 + C\alpha k\right)^n.$$

Taking into account that $(1 + C\alpha k)^n \leq (1 + C\alpha k)^{T_M/k}$, and that $(1 + C\alpha k)^{T_M/k}$ increases with k decreasing, with

$$\lim_{k\to 0}(1 + C\alpha k)^{T_M/k} = e^{T_M C\alpha},$$

it follows that,

$$\|Q_k^n - Q^n\| \leq C^2 \alpha T_M e^{T_M C\alpha},$$

for all $n \geq 0$. Thus, for $0 < k < k_0$ and $0 \leq nk \leq T_M$, it holds that the matrices Q_k^n are uniformly bounded,

$$\|Q_k^n\| = \|Q_k^n - Q^n + Q^n\| \leq C^2 \alpha T_M e^{T_M C\alpha} + C,$$

and hence the scheme is stable, unconditionally, in the sense of Lax [22].

4. Truncation errors, consistency, and convergence

Next, bounds on the truncations errors of the scheme will be obtained, and thus, assuming sufficient smoothness of the solution, the consistency of the method will follow, which, together with the results of the last section, will assure its convergence. In what follows, the exact solution of the problem (11)-(14) will be denoted $w = \left(\widetilde{u}, \widetilde{T}, \widetilde{v}\right)$.

Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ be the difference operator associated with the scheme [24, p. 42], where

$$\begin{aligned}
\mathcal{L}_1 w_j^n &= \frac{1}{10k}\left(\widetilde{u}_{j-1}^{n+1} - \widetilde{u}_{j-1}^n + 10\left(\widetilde{u}_j^{n+1} - \widetilde{u}_j^n\right) + \widetilde{u}_{j+1}^{n+1} - \widetilde{u}_{j+1}^n\right) \\
&\quad - \frac{3}{15}\left(a\delta_x^2\left(\widetilde{T}_j^{n+1} + \widetilde{T}_j^n\right) + b\delta_x^2\left(\widetilde{v}_j^{n+1} + \widetilde{v}_j^n\right) + c\delta_x^2\left(\widetilde{u}_j^{n+1} + \widetilde{u}_j^n\right)\right) \\
\mathcal{L}_2 w_j^n &= \frac{1}{k}\left(\widetilde{T}_j^{n+1} - \widetilde{T}_j^n\right) + \frac{B}{2C}\left(\widetilde{T}_j^{n+1} + \widetilde{T}_j^n\right) - \frac{1}{2C}\left(\widetilde{v}_j^{n+1} + \widetilde{v}_j^n\right) \\
\mathcal{L}_3 w_j^n &= \frac{1}{k}\left(\widetilde{v}_j^{n+1} - \widetilde{v}_j^n\right) - \frac{1}{2}\left(\widetilde{u}_j^{n+1} + \widetilde{u}_j^n\right) + \frac{A}{2}\left(\widetilde{T}_j^{n+1} + \widetilde{T}_j^n\right),
\end{aligned}$$

so that the local truncation error at the point (jh, nk), [22, p. 20], is given by $\left(\mathcal{T}_{1,j}^n, \mathcal{T}_{2,j}^n, \mathcal{T}_{3,j}^n\right) = \mathcal{L}w_j^n$.

Using appropriate Taylor series expansions, it is not difficult to show that the following bounds hold,

$$\mathcal{T}_{1,j}^n = O\left(k^2\right) + O\left(h^4\right), \ \mathcal{T}_{2,j}^n = O\left(k^2\right), \ \mathcal{T}_{3,j}^n = O\left(k^2\right). \tag{32}$$

Considering the vectors of exact solutions at the points in the mesh,

$$\widetilde{U}^n = \left[\widetilde{u}_1^n, \widetilde{u}_2^n, \ldots, \widetilde{u}_{P-1}^n\right]^T, \ \widetilde{T}^n = \left[\widetilde{T}_1^n, \widetilde{T}_2^n, \ldots, \widetilde{T}_{P-1}^n\right]^T, \ \widetilde{V}^n = \left[\widetilde{v}_1^n, \widetilde{v}_2^n, \ldots, \widetilde{v}_{P-1}^n\right]^T.$$

and the stack vector $W^n = \left[\widetilde{U}^n, \widetilde{T}^n, \widetilde{V}^n\right]^T \in \mathbb{R}^{3(P-1)}$, the truncation error of the two-level scheme (30), at $t = nk$, is the vector

$$\mathfrak{T}^n = \frac{1}{k}\left(W^{n+1} - Q_k W^n\right), \tag{33}$$

and one can verify that

$$\mathfrak{T}^n = E_k \left[\mathcal{T}_1^n, \mathcal{T}_2^n, \mathcal{T}_3^n\right]^T, \tag{34}$$

where $\mathcal{T}_i^n = \left[\mathcal{T}_{i,1}^n, \mathcal{T}_{i,2}^n, \ldots, \mathcal{T}_{i,P-1}^n\right]^T$, $i = 1, 2, 3$, and

$$E_k = \begin{bmatrix} 10X_k^{-1}S^{-1} & \frac{12rC}{x_k}\left(2a - bAk\right)X_k^{-1}S^{-1}M & E_k[1,3] \\ 10\frac{k^2}{x_k}X_k^{-1}S^{-1} & \frac{4C}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k - \frac{k^2 r}{2x_k}\left(2a - bAk\right)\right)S^{-1}M\right) & E_k[2,3] \\ 10\frac{(2C+Bk)k}{x_k}X_k^{-1}S^{-1} & -\frac{2ACk}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k + r\frac{2C+Bk}{Ax_k}\left(2a - bAk\right)\right)S^{-1}M\right) & E_k[3,3] \end{bmatrix},$$

with

$$E_k[1,3] = \frac{12r}{x_k}\left(2bC + \left(Bb + a\right)k\right)X_k^{-1}S^{-1}M,$$

$$E_k[2,3] = \frac{2k}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k - \frac{kr}{x_k}\left(2bC + \left(Bb + a\right)k\right)\right)S^{-1}M\right),$$

and

$$E_k[3,3] = \frac{2\left(2C + Bk\right)}{x_k}X_k^{-1}\left(I - 6\left(\alpha_k - \frac{kr}{x_k}\left(2bC + \left(Bb + a\right)k\right)\right)S^{-1}M\right).$$

It is not difficult to verify that the matrix $E = \lim_{k\to 0} E_k$ is given by

$$E = \begin{bmatrix} 10X^{-1}S^{-1} & 6raX^{-1}S^{-1}M & 6rbX^{-1}S^{-1}M \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{35}$$

and hence it follows that E_k has bounded norm. Therefore, from (32) and (34), it holds that, for $0 \le nk \le T_M$,

$$\lim_{k\to 0} \frac{1}{k} \left\| W_h^k\left((n+1)\,k\right) - Q_k W_h^k\left(nk\right) \right\|_\infty = 0, \tag{36}$$

so that the scheme is consistent with the partial differential equation. Also, it can be deduced that the truncation error satisfies

$$\|\mathfrak{T}^n\|_\infty = O\left(k^2\right) + O\left(h^4\right). \tag{37}$$

As a result, since the method is unconditionally stable, for any finite value of $r = h/k^2$, and it is consistent with the original partial differential equation, from Lax equivalence theorem [22] it follows that the method is also unconditionally convergent of order $O\left(k^2\right) + O\left(h^4\right)$.

5. Numerical examples

In this section, two examples of application of the new scheme to DPL(2,2) models are presented, comparing its accuracy with that of the previous method in [20]. The selected problems include simple initial functions, allowing for the computation of exact solutions, so that the errors of the numerical approximations provided by the scheme, with respect to the exact values, can be illustrated.

Consider the problem (5)-(7), with initial conditions

$$T(x,0) = \sin(\pi x),\ \frac{\partial}{\partial t}T(x,0) = -\alpha_{Cu}\pi^2\sin(\pi x),\ \frac{\partial^2}{\partial t^2}T(x,0) = 0,\ x \in [0,l]\,,$$

and with parameters $l = 1$, and α, τ_q, and τ_T corresponding to those experimentally found in copper [7, p. 123], $\alpha = 1.1283 \cdot 10^{-4}$ m^2/s, $\tau_q = 0.4648$ ps, and $\tau_T = 70.833$ ps.

In Figure 1, the absolute differences between the exact solution and the numerical approximations,

$$\left| \widetilde{T}\,(x,t) - T\,(x,t) \right|,$$

obtained either with the new scheme, left subfigures, or with the previous method in [20], right subfigures, are presented, using two different mesh sizes in the same domain.

A more detailed illustration of the error properties of the new scheme, and its better accuracy as compared with the method in [20], is presented in Figures 2 and 3, for a similar DPL(2,2) model but with initial conditions

$$T(x,0) = \sin(\pi x),\ \frac{\partial}{\partial t}T(x,0) = -\pi^2 \sin(\pi x),\ \frac{\partial^2}{\partial t^2}T(x,0) = \pi^4 \sin(\pi x),\ x \in [0,l]\,,$$

and with parameters $l = 1$, $\alpha = 1$, $\tau_q = \frac{1}{16}$, and $\tau_T = \frac{2}{\pi^2} - \frac{1}{16}$.

In Fig. 2, left, the maximum absolute errors of the approximate solution obtained with the new method as a function of time is presented, for three different mesh sizes. The right subfigure shows the relative errors in relation with the mesh size, in agreement with the order of the method.

In Fig. 3, the absolute errors in Fig. 2 (left) are compared with the corresponding errors of the previous method in [20], showing the higher accuracy of the new scheme.

Acknowledgements

This work was partially funded by grant GRE12-08 from University of Alicante.

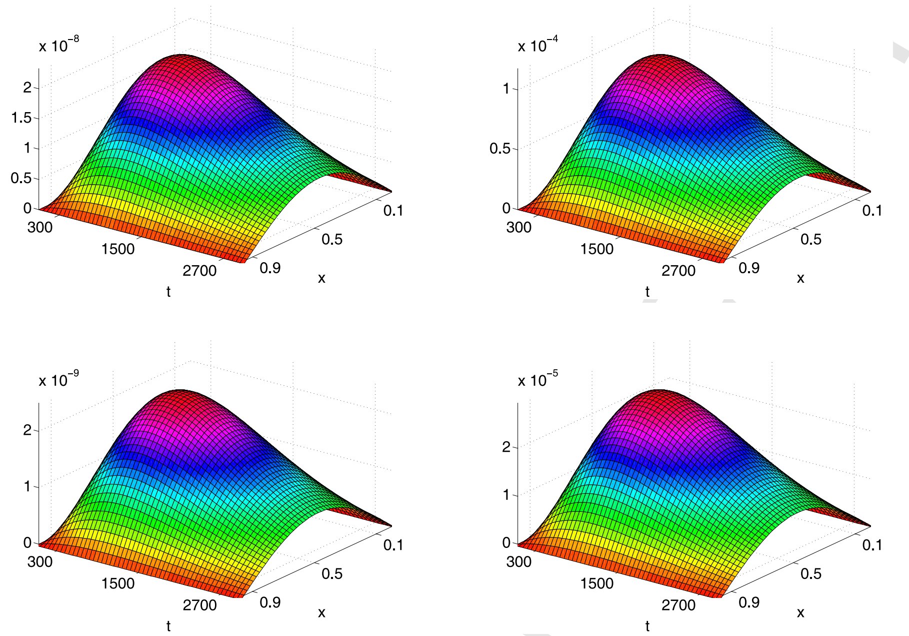

Figure 1: Absolute errors, $\left|\widetilde{T}\,(x,t) - T\,(x,t)\right|$, with respect to the exact values $\widetilde{T}(x,t)$, for the numerical approximations of DPL(2,2) model, $T(x,t)$, obtained with the scheme of this work (left), and with the method in [20] (right), for two different mesh sizes (top: $\Delta x = \Delta t = 0.02$; bottom: $\Delta x = \Delta t = 0.01$).

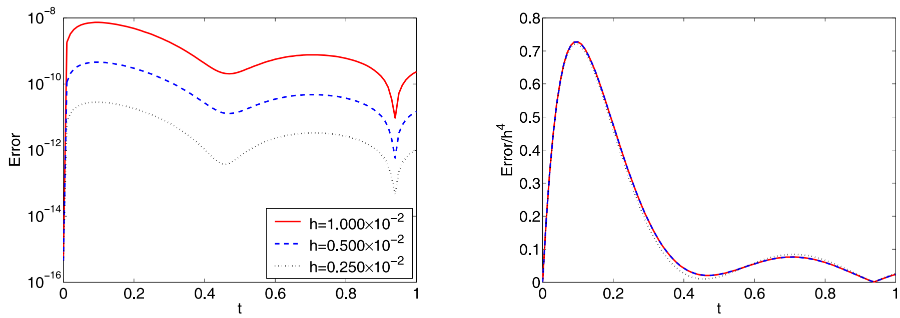

Figure 2: Maximum errors of the approximate solutions, in terms of time, for three different mesh sizes (left), and relation with the size of the mesh (right).

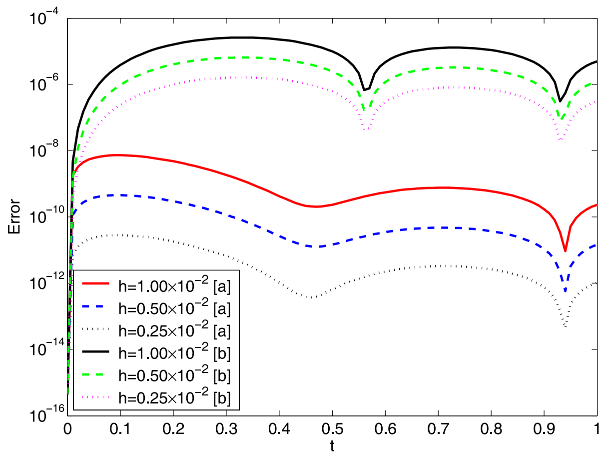

Figure 3: Comparison of the absolute errors for the new scheme developed in this work and the previous method in [20], for three different mesh sizes. New scheme: lower three lines, [a] (as in Fig. 2 left). Previous method: upper three lines, [b].

References

[1] L. Wang, X. Wei, Heat conduction in nanofluids, Chaos Solitons Fractals 39 (2009) 2211-2215.

[2] J.J. Vadasz, S. Govender, Thermal wave effects on heat transfer enhancement in nanofluids suspensions, Int. J. Thermal Sci. 49 (2010) 235-242.

[3] T. Q. Qiu, C. L. Tien, Heat transfer mechanisms during short-pulse laser heating of metals, ASME J. Heat Transfer 115 (1993) 835-841.

[4] F. Xu, K.A. Seffen, T.J. Liu, Non-Fourier analysis of skin biothermomechanics, Int. J. Heat Mass Transfer 51 (2008) 2237-2259.

[5] J. Zhou, J.K. Chen, Y. Zhang, Dual-phase-lag effects on thermal dam-

age to biological tissues caused by laser irradiations, Comput. Biol. Med., 39 (2009) 286-293.

[6] D.D. Joseph, L. Preziosi, Heat waves, Rev. Mod. Phys. 61 (1989) 41-73.

[7] D.Y. Tzou, Macro- to Microscale Heat Transfer: The Lagging Behavior, Taylor & Francis, Washington, 1996.

[8] D.Y. Tzou, The generalized lagging response in small-scale and high-rate heating, Int. J. Heat Mass Transfer 38 (1995) 3231-3240.

[9] D.Y. Tzou, Experimental support for the lagging behavior in heat propagation, AIAA J. Termophys. Heat Transfer 9 (1995) 686-693.

[10] V.V. Kulish, V.B. Novozhilov, An integral equation for the dual-lag model of heat transfer, ASME J. Heat Transfer 126 (2004) 805-808.

[11] M. Xu and L. Wang, Dual-phase-lagging heat conduction based on Boltzmann transport equation, Int. J. Heat Mass Transfer 48 (2005) 5616-5624.

[12] D.Y. Tzou, A unified approach for heat conduction from macro to micro-scales, ASME J. Heat Transfer 117 (1995) 8-16.

[13] R. Quintanilla, R. Racke, A note on stability in dual-phase-lag heat conduction, Int. J. Heat Mass Transfer 49 (2006) 1209-1213.

[14] W. Dai, R. Nassar, A finite difference scheme for solving the heat transport equation at the microscale, Numer. Methods Partial Differential Equations 15 (1999) 697-708.

[15] W. Dai, R. Nassar, A compact finite difference scheme for solving a three-dimensional heat transport equation in a thin film, Numer. Methods Partial Differential Equations 16 (2000) 441-458.

[16] W. Dai, R. Nassar, A finite difference scheme for solving a theedimensional heat transport equation in a thin film with microscale thickness, Int. J. Numer. Methods Biomed. Eng. 50 (2001) 1665-1680.

[17] W. Dai, R. Nassar, A compact finite difference scheme for solving a one-dimensional heat transport equation at the microscale, J. Comput. Appl. Math. 132 (2001) 431-441.

[18] W. Dai, R. Nassar, An unconditionally stable finite difference scheme for solving a 3D heat transport equation in a sub-microscale thin film, J. Comput. Appl. Math. 145 (2002) 247-260.

[19] H. Wanng, W. Dai, R. Nassar, R. Melnik, A finite difference method for studying thermal deformation in a thin film exposed to ultrashort-pulsed lasers, Int. J. Heat Mass Transfer 49 (2006) 2712-2723.

[20] J. Cabrera, M.A. Castro, F. Rodríguez, and J.A. Martín, Difference schemes for numerical solutions of lagging models of heat conduction, Math. Comput. Modelling 57 (2013) 1625-1632.

[21] S.K. Lele, Compact finite difference schemes with spectral-like resolution, J. Comput. Phys. 103 (1992) 1642.

[22] R.D. Richtmyer, K.W. Morton, Difference Methods for Initial-Value Problems, John Wiley & Sons, New York (1967).

[23] R.P. Agarwal, Difference Equations and Inequalities: Theory, methods and applications, Second Edition, Marcel Dekker, Inc., New York (2000).

[24] J.W. Thomas, Numerical Partial Differential Equations, Springer-Verlag, New York (1995).